# A pairing between G-equivariant vector bundles and G-equivariant Dirac operators.

Let $E \in \mathrm{Vect}_G(X)$ be a G-equivariant vector bundles with a hermitian connection $\nabla_E$ and $(S, c)$ be a G-Clifford symbol for $TX$.

Then $(S \otimes E, c \otimes \mathrm{id}_E)$ with a hermitian connection

$\nabla_{S\otimes E} := \nabla_S \otimes \mathrm{id}_E + \mathrm{id}_S \otimes \nabla_E$ is another Clifford symbol for $TX$. This leads to a new Dirac operator, denoted by $D_E$. One calls it <u>a twisted Dirac op.</u>

This defines the map

$$K_G^0(X) \longrightarrow R(G), \quad [E] \longmapsto \mathrm{ind}_G(D_E).$$

This can be used to prove the following

**Proposition.** Let $X = G/T$, where $G = SU(n)$, $T$ the diagonal torus, be the complex full flag variety. Then the restriction map

$$R(G) \longrightarrow R(T)$$

induces an isomorphism

$$R(G) \xrightarrow{\cong} R(T)^W$$

onto the Weyl group-invariant subspace, where $W = N_G(T)/T$.

**Proof.** (sketch of)

**Lemma.** Let $G$ be a compact connected Lie group and $T$ a maximal torus in $G$. Then

$$q : G/_T \times T \longrightarrow G \quad , \quad (gT, t) \longmapsto g t g^{-1}$$

is a finite map of degree $|W|$, in particular $q$ is surjective.

**Proof.** Coadjoint representation:

$$Ad : G \longrightarrow Aut(\mathfrak{g}) \quad , \quad g \longmapsto d_e c(g)$$

where $c(g) : g' \longmapsto g g' g^{-1}$

$G$ compact $\Rightarrow$ $\exists$ Ad-invariant left invariant volume form $v_G$ on $G$ and an Ad-invariant decomposition $\mathfrak{g} = \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$, $Ad_T : \mathfrak{t} \to \mathfrak{t}$ trivial, $Ad_T|_{\mathfrak{g}/\mathfrak{t}} : \mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathfrak{t}$ nontrivial on every nonzero direction since $T$ is a maximal torus.

(Hence its representation makes $\mathfrak{g}/\mathfrak{t}$ a direct sum of complex planes corresponding to rotations, since $\mathfrak{t}$ is a maximal subspace of $\mathfrak{g}$ acted on trivially by $Ad_T$)

$\pi : G \to G/T$ induces a map $\mathfrak{g} \longrightarrow T_{eT}(G/T) \cong \mathfrak{g}/\mathfrak{t}$

s.t. $\quad \nu_G = \pi^* \nu_{G/T} \wedge \omega, \quad \omega|_T = \nu_T$.

We prove that for $\det(q) := \left( (q^* \nu_G) / p_1^* \nu_{G/T} \wedge p_2^* \nu_T \right)$

$$\det(q)(gT, t) = \det\left( Ad_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}} - Id_{\mathfrak{g}/\mathfrak{t}} \right)$$

where

$$\begin{tikzcd} & G/T \times T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G/T & & T \end{tikzcd}$$

are canonical projections in the cartesian product.

Since $G$ acts transitively on $G$ and $G/T$, and $T$ acts transitively on $T$, and $\nu_G$, $\nu_{G/T}$ and $\nu_T$ are left invariant it is enough to compute the left hand side at $(eT, e) \in G/T \times T$ and then translate as follows

$$\begin{tikzcd}
G/T \times T \arrow[r, dashed] \arrow[d, "L_{(g,t)}"'] & G \\
G/T \times T \arrow[r, "q"] & G \arrow[u, "(L_{gtg^{-1}})^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(g'T, t') \arrow[r, mapsto, dashed] \arrow[d, mapsto] & c(g)\left(c(t^{-1})(g')\,t'\,(g')^{-1}\right) \\
(gg'T, tt') \arrow[r, mapsto] & (gg')(tt')(gg')^{-1} \arrow[u, mapsto]
\end{tikzcd} \qquad (*)$$

In particular for $g' = e$, $t' = e$

$$\begin{array}{ccc} (eT, e) & \longmapsto & c(g)\, c(t^{-1})(e) \\ \downarrow & & \uparrow \\ (gT, t) & \longmapsto & c(g)t \end{array}$$

Thus the determinant of the derivative is the determinant of

$$\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t} \longrightarrow \mathfrak{g} \cong \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$$

$$(X+\mathfrak{t}, Y) \longmapsto \mathrm{Ad}_{t^{-1}}\big|_{\mathfrak{g}/\mathfrak{t}}(X) + Y - X$$

(by the Leibniz rule after a faithful matrix representation of (X) and the fact that $\det \mathrm{Ad}_g = 1$, since $\nu_G$, $\nu_{G/T}$ are $G$-invariant and $G$ is connected), of the block form

$$\left[\begin{array}{c|c} \mathrm{Ad}_{t^{-1}}\big|_{\mathfrak{g}/\mathfrak{t}} - \mathrm{Id}_{\mathfrak{g}/\mathfrak{t}} & 0 \\ \hline 0 & \mathrm{Id}_{\mathfrak{t}} \end{array}\right]. \qquad \blacksquare$$

Now it suffices to count the pre-images of any regular value of $q$ and to keep track of orientation, as follows.

**Lemma.** Let $t'$ be an element of $T$ generating a dense subgroup (after Kronecker). Then

i) $\# q^{-1}(t') = |W|$

ii) $\det(q) > 0$ at all preimages of $t'$.

**Proof.** i) $q(gT, t) = t' \iff g t g^{-1} = t' \iff t = g^{-1} t' g \in T$

such $t$ exists iff $g^{-1} T g \subset T \Rightarrow g \in N_G(T)$

$$\Rightarrow \quad q^{-1}(t') = \{ (gT, g^{-1} t' g) \mid g \in N_G(T) \}$$

$$\Rightarrow \quad \# q^{-1}(t') = \# N_G(T)/T = |W| .$$

ii) It suffices to prove that $Ad_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}} - Id_{\mathfrak{g}/\mathfrak{t}}$ has no real eigenvalues, then the complex eigenvalues come in complex conjugate pairs, hence the determinant is positive. But $Ad_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}}$ moves every nonzero real direction, so it has no real eigenvalues, hence it is so after subtracting the identity linear map. $\square$

Note that since the degree is constant locally around a regular value, $e \in T$ is a regular value (at which $d_e q$ is equivalent to the block matrix

$$\left[\begin{array}{c|c} Ad_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}} - Id_{\mathfrak{g}/\mathfrak{t}} & 0 \\ \hline 0 & Id_{\mathfrak{t}} \end{array}\right]$$

which is invertible since $0$ is not an eigenvalue as above), one can assume that the Kronecker generator taken in sufficiently small neighborhood of $e \in T$ is a regular value of $q$.

Therefore $q : G/T \times T \longrightarrow G$ is a surjective finite map of degree $|W|$. $\square$

Now, let $T$ and $T'$ be maximal tori and $t' \in T'$ be a Kronecker generator. By the Lemma

$\exists g \in G \quad t' \in gTg^{-1} \Longrightarrow T' \subset gTg^{-1}$ and since $gTg^{-1}$ is a torus and $T'$ is a maximal torus $T' = gTg^{-1}$. Since $q$ is surjective every $g \in G$ is contained in some conjugate of $T$.

This is an important fact, we use it to prove the next lemma.

**Lemma.** The action of $W = W_G(T)$ on $T$ is effective and two elements of $T$ are conjugate in $G$ iff they lie in the same orbit of $W$.

**Proof.** $\mathrm{Ker}\,(W \longrightarrow \mathrm{Aut}(T)) = Z_G(T)/T$.

We prove that $Z_G(T) = T$, hence $W \longrightarrow \mathrm{Aut}(T)$ is injective. Let $z \in Z_G(T)$, and let $H$ be the closure of the subgroup generated by $z$ and $T$. Then $H$ is compact and abelian, so its connected component $H_0$ is a connected abelian compact subgroup so it is a torus. Since $T \subset H_0$ and

$T$ is a maximal torus, $H_0 = T$. Since $xT$ generates $H/T$, $H/T$ is a finite cyclic group, $H/T \cong \mathbb{Z}/m\mathbb{Z}$. Since the abelian group extension

$$1 \to T \to H \to \mathbb{Z}/m\mathbb{Z} \to 1$$

is split (any lifting of the generator gives a splitting), $H \cong T \times \mathbb{Z}/m\mathbb{Z}$. The Kronecker generator $t'$ of $T$ can be chosen in such a way that $t' \in T$ and the generator of $\mathbb{Z}/m\mathbb{Z}$ form a pair $g \in T \times \mathbb{Z}/m\mathbb{Z}$ such that $g$ generates a dense subgroup of $H$. But $g$ is contained in some maximal torus $T'$, so $T \cup \{z\} \subset T'$. But $T$ is a maximal torus

$\Rightarrow T = T' \Rightarrow z \in T \Rightarrow Z_G(T) = T$.

Now, let $t_1, t_2 \in T$, $g \in G$, $g t_1 g^{-1} = t_2$,

$Z_G(t_1), Z_G(t_2)$ centralizers, $c(g) : Z_G(t_1) \longrightarrow Z_G(t_2)$

conjugation $c(g)(z_1) = g z_1 g^{-1}$.

$T \subset Z_G(t_1) \Longrightarrow c(g) T \subset Z_G(t_2)$

$\Rightarrow$ $T$ and $c(g) T$ maximal tori in the connected component $Z_G(t_2)_0$ of $Z_G(t_2)$

$\Rightarrow \exists\, z \in Z_G(t_2)_0$ with $T = c(z) c(g) T = c(zg) T$

$\Rightarrow zg \in N_G(T)$ and $c(zg) t_1 = c(z) t_2 = t_2$

$\Rightarrow w := zgT \in N_G(T)/T = W_G(T)$ and $w t_1 = t_2$. $\blacksquare$

**Lemma.** Let $\mathrm{Con}(G)$ be the space of conjugacy classes of $G$ with the quotient topology with respect to $G \times G \longrightarrow G$, $(g, g') \longmapsto g g' g^{-1}$. There is a canonical homeomorphism

$$\kappa : W \backslash T \xrightarrow{\cong} \mathrm{Con}(G) \, , \quad Wt \longmapsto c(G)t \, .$$

**Proof.** $\kappa$ well defined and continuous. It is surjective because conjugates of $T$ cover $G$, and injective by the previous lemma.
Both spaces are compact Hausdorff, so $\kappa$ is a homeomorphism. $\square$

Therefore, $C(\mathrm{Con}(G)) \cong C(W\backslash T) \cong C(T)^W$

where for $B := C(T)$ $\quad (bw)(t) := b(wt)$.

and $A := C(\mathrm{Con}(G)) \longrightarrow C(T)$ is $a \longmapsto a|_T$.

Characters of representations are class form a subring $R(G) \subset C(\mathrm{Con}(G))$,

so the embedding $T \hookrightarrow G$ induces the restriction map

$$R(G) \longrightarrow R(T)^W, \quad \chi \longmapsto \chi|_T$$

which is injective.

The inverse $R(T)^W \to R(G)$ can be constructed as follows. Let $[V] \in R(T)$ be a class of a representation of $T$, and $E := G \times^T V \in \mathrm{Vect}_G(G/T)$. $G/T = G^{\mathbb{C}}/B$

$\Rightarrow$ $G/T$ complex manifold, equipped with the Cauchy-Riemann operator

$$\bar{\partial} : \bigoplus_{q \text{ even}} \Gamma(\Lambda^{0,q} T^*) \longrightarrow \bigoplus_{q \text{ odd}} \Gamma(\Lambda^{0,q+1} T^*)$$

Then $\mathrm{ind}_G((\bar\partial + \bar\partial^*)_E) \in R(G)$.

This gives a map $R(T) \longrightarrow R(G)$. $(*)$

Restricting $(*)$ to $W$-invariants $R(T)^W$

we obtain a map $R(T)^W \longrightarrow R(G)$.

One proves that if $[V_0]-[V_1] \in R(T)^W$ and $E_i := G\times^T V_i$,

$\mathrm{ind}_G((\bar\partial+\bar\partial^*)_{E_0}) - \mathrm{ind}_G((\bar\partial+\bar\partial^*)_{E_1}) = [W_0]-[W_1] \in R(G)$

then $[W_0|_T] - [W_1|_T] = [V_0]-[V_1]$

(Hodge theory + Borel-Weyl-Bott theorem + highest weight theory + ...)

This proves surjectivity of $R(G) \longrightarrow R(T)^W$. $\square$

**Exercise 32.** Show that the Weyl group acts on $K_G(G/T)$.

**Solution.** If $w = nT \in W$, $n \in N_G(T)$ then

$$(gT)w := gnT$$

defines a <u>right</u> action (hence commuting with the <u>left</u> $G$-action on $G/T$)

$$(G/T) \times W \longrightarrow G/T, \qquad (gT, w) \longmapsto (gT)w$$

and hence by functoriality of $K_G(G/T)$ an action

$$K_G(G/T) \times W \longrightarrow K_G(G/T). \quad \blacksquare$$